Int. J. Solids Structures Vol. 21, No. 4, pp. 343-353, 1985
Printed in the U.S.A.

0020-7683/85 $3.00 + .00


# EXTENSIONAL WAVES IN STRONGLY ANISOTROPIC ELASTIC PLATES

W. A. GREEN
Department of Theoretical Mechanics, University of Nottingham, Nottingham, England

DRAGAN MILOSAVLJEVIC
Faculty of Mechanical Engineering, University of Svetozar Markovic, Kragujevac, Yugoslavia

(*Received* 7 *September* 1982)

**Abstract**—This article examines extensional waves propagating in an infinite plate of transversely isotropic material with the axis of transverse isotropy lying in the plane of the plate. Attention is restricted to waves propagating in the direction of transverse isotropy and an equation is derived for the phase velocity as a function of wavelength. Expressions are also obtained for the variation of stress through the thickness of the plate. Corresponding results are obtained for the idealized material which is inextensible in the direction of transverse isotropy. These two materials provide possible models for an elastic matrix reinforced by a family of parallel fibres. A comparison of the results shows that the inextensible material behaves quite differently from the anisotropic material in the long-wavelength region.

## 1. INTRODUCTION

The macroscopic behaviour of a composite material consisting of an isotropic elastic matrix reinforced by a family of parallel elastic fibres can be modelled by a continuum theory which treats the composite as a transversely isotropic elastic material with the axis of transverse isotropy coinciding with the fibre direction (see, for example, Spencer[1]). Generally the extensional modulus of the fibres is considerably higher than a typical modulus of the matrix and the macroscopic model will be strongly anisotropic, with the extensional modulus in the direction of transverse isotropy being very much greater than that in any direction at right angles to the fibres. An idealization of this behaviour is to treat the continuum as being inextensible in the fibre direction and this leads to a comparatively simple system of governing equations for static boundary value problems. A number of solutions of these equations have been obtained by England, Ferrier and Thomas[2] and by Morland[3]. Pipkin[4] has shown that this inextensible theory is an asymptotic approximation to the theory of transversely isotropic elasticity in the limit as the extensibility in the fibre direction tends to zero. Both Pipkin[4] and Spencer[1] discuss the role of the solutions to static problems in the inextensible theory in providing information about solutions to the corresponding problems for strongly anisotropic materials.

In an attempt to examine the relation between the inextensible and strongly anisotropic models in dynamic problems, Green[5] has considered the propagation of flexural waves in an infinite plate using the two theories. The conclusion is that the inextensible model fails to reproduce the long-wave (low-frequency) behaviour of the strongly anisotropic material. In this paper we extend this investigation to the consideration of extensional waves in an infinite plate, restricting consideration to waves propagating in the fibre direction. Solutions in the inextensible theory are derived in Section 2; Section 3 is devoted to longitudinal waves in a plate of transversely isotropic material. Approximate forms of the solutions of Section 3 for strongly anisotropic materials are obtained in Section 4 and the limiting case of zero extensibility in the fibre direction is examined. Section 5 contains numerical results for the wave velocity as a function of wavelength for a particular composite material and graphs of the stress variation through the plate thickness at a number of discrete wavelengths. The final section is devoted to a discussion of the results.

## 2. INEXTENSIBLE PLATE

We consider an infinite plate of transversely isotropic elastic material with the direction of transverse isotropy lying in the plane of the plate. Choosing cartesian coordinates $Ox_1$ normal to the plate, $Ox_2$ and $Ox_3$ in the middle plane of the plate with $Ox_3$ in the direction of transverse isotropy, the boundary surfaces of the plate are given by $x_1 = \pm h$, where $2h$ is the plate thickness. We restrict attention to longitudinal waves propagating in the $x_3$-direction, corresponding to a state of plane strain in the $x_1x_3$ plane, and in this section we assume the material to be inextensible in the direction of transverse isotropy. The constitutive equations in plane strain are then (see Spencer[1])

$$t_{11} = (\lambda + 2\mu_T)e_{11},\ t_{22} = \lambda e_{11},\ t_{33} = \hat{T}(x_k,t),\ t_{13} = 2\mu_L e_{13}, \tag{2.1}$$

to which must be added the inextensibility constraint

$$e_{33} = 0. \tag{2.2}$$

In eqns (2.1) and (2.2), $t$ is the time, $\lambda$, $\mu_T$ and $\mu_L$ are material constants, $t_{ij}$ are the components of the Cauchy stress tensor, $\hat{T}$ is a reaction stress associated with the inextensibility constraint and $e_{ij}$ are components of the strain tensor defined in terms of the displacement components $u_i(x_k,t)$ by the relations

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{2.3}$$

For plane waves propagating with speed $v$ and wave number $k$ in the $x_3$-directions the displacements $u_i$ and reaction stress $\hat{T}$ have the form

$$u_1 = U(x_1)\cos\phi,\ u_2 = 0,\ u_3 = W(x_1)\sin\phi,\ \hat{T} = T(x_1)\cos\phi, \tag{2.4}$$

where

$$\phi = k(x_3 - vt). \tag{2.5}$$

The inextensibility constraint (2.2) leads immediately to

$$W = 0 \tag{2.6}$$

and eqns (2.4) when substituted into (2.1) give

$$\begin{aligned} t_{11} &= (\lambda + 2\mu_T)U'\cos\phi, & t_{22} &= \lambda U'\cos\phi, \\ t_{33} &= T\cos\phi, & t_{13} &= -\mu_L kU\sin\phi, \end{aligned} \tag{2.7}$$

where the prime denotes differentiation with respect to $x_1$.

The equations of motion in the $x_1x_3$ plane are

$$\begin{aligned} \frac{\partial t_{11}}{\partial x_1} + \frac{\partial t_{13}}{\partial x_3} &= \rho\ddot{u}_1, \\ \frac{\partial t_{31}}{\partial x_1} + \frac{\partial t_{33}}{\partial x_3} &= \rho\ddot{u}_3, \end{aligned} \tag{2.8}$$

where $\rho$ is the density of the material and the dots denote differentiation with respect to time. Equations (2.8) are to be solved subject to traction-free boundary conditions on the upper and lower surfaces of the plate, namely

$$t_{11} = 0, \quad t_{13} = 0 \quad \text{at} \quad x_1 = \pm h. \tag{2.9}$$

Substituting from (2.4), (2.6) and (2.7) into the equations of motion (2.8) gives the equations

$$c_1^2 U'' + k^2(v^2 - c_3^2)U = 0, \qquad c_3^2 U' + \frac{T}{\rho} = 0, \tag{2.10}$$

where

$$c_1^2 = \frac{\lambda + 2\mu_T}{\rho}, \qquad c_3^2 = \frac{\mu_L}{\rho}. \tag{2.11}$$

Equations (2.10) are to be solved for $U(x_1)$, $T(x_1)$ subject to the boundary conditions

$$U' = 0, \qquad U = 0 \quad \text{at} \quad x_1 = \pm h, \tag{2.12}$$

which arise from substituting (2.7) into (2.9). The general solution of eqns (2.10) is

$$U = A \sin kpx_1 + B \cos kpx_1, \quad T = -\rho c_3^2 kp(A \cos kpx_1 - B \sin kpx_1), \tag{2.13}$$

where $A$ and $B$ are arbitrary constants and

$$p^2 = \frac{v^2 - c_3^2}{c_1^2}. \tag{2.14}$$

The boundary conditions (2.12) are satisfied by the solutions (2.13) provided

$$\begin{aligned} A \sin kph &= 0, \qquad pA \cos kph = 0, \\ B \cos kph &= 0, \qquad pB \sin kph = 0. \end{aligned} \tag{2.15}$$

Equations (2.15) have only the trivial solution $A = B = 0$ or the solution $B = 0$, $p = 0$. The latter corresponds to $v^2 = c_3^2$, for which the general solution of eqns (2.10) is no longer given by (2.13) but has the form

$$U = \bar{A}x_1 + \bar{B}, \qquad T = -\rho c_3^2 \bar{A}, \tag{2.16}$$

where $\bar{A}$ and $\bar{B}$ are arbitrary constants. The solutions (2.16) satisfy the boundary conditions (2.12) only if $\bar{A} = 0$, $\bar{B} = 0$, which again gives the trivial solution.

In order to get non-trivial solutions it is necessary to postulate the existence of singular layers at the upper and lower surfaces of the plate (see Green[5]). In the singular layers the reaction stress $\hat{T}$ becomes infinite, corresponding to finite loads $\hat{L}^+$ and $\hat{L}^-$ in the upper and lower surfaces, respectively. Across a singular layer, the shear stress $t_{13}$ is discontinuous, being zero at $x_1 = \pm h$ but having a non-zero limit as $x_1 \to \pm h$ from the interior of the plate. That such singular surfaces are possible may be deduced from integrating the second of the equations of motion (2.8) from $x_1 = h(1 - \sigma)$ to $x_1 = h$ for any $\sigma$ satisfying $0 < \sigma < 1$. This gives

$$[t_{31}]_{x_1 = h(1-\sigma)}^{x_1 = h} + \frac{\partial}{\partial x_3}\int_{h(1-\sigma)}^{h} t_{33}\,dx_1 = \int_{h(1-\sigma)}^{h} \rho \ddot{u}_3\,dx_1, \tag{2.17}$$

which on using (2.6), (2.7) and the condition that $t_{31} = 0$ at $x_1 = h$, reduces to

$$\mu_L U\{h(1 - \sigma)\} - \int_{h(1-\sigma)}^{h} T(x_1)\,dx_1 = 0, \tag{2.18}$$

where we have eliminated the factor $k \sin \phi$. Proceeding to the limit as $\sigma \to 0$, eqn

(2.18) yields

$$L^+ = \lim_{\sigma \to 0} \int_{h(1-\sigma)}^{h} T(x_1)\, dx_1 = \rho c_3^2 U(h), \tag{2.19}$$

and the load $\hat{L}^+$ in the upper surface is given by $\hat{L}^+ = L^+ \cos\phi$. Repeating this process from $x_1 = -h(1 - \sigma)$ to $x_1 = -h$ gives, at the lower surface,

$$L^- = -\lim_{\sigma \to 0} \int_{-h(1-\sigma)}^{-h} T(x_1)\, dx_1 = -\rho c_3^2 U(-h), \tag{2.20}$$

with a corresponding load $\hat{L}^- = L^- \cos\phi$.

With the boundary conditions $t_{13} = 0$ at $x_1 = \pm h$ accounted for by these singular surfaces, it is necessary to solve eqn (2.10) subject only to the boundary condition

$$U' = 0 \text{ at } x_1 = \pm h, \tag{2.21}$$

corresponding to the traction condition $t_{11} = 0$. Using the solutions (2.13), eqn (2.21) is satisfied provided

$$pA \cos kph = 0, \qquad pB \sin kph = 0. \tag{2.22}$$

The non-trivial solutions to these are

$$\begin{aligned} A &= 0, \quad kph = n\pi, & n &= 0,1,2 \ldots \text{ and} \\ B &= 0, \quad kph = (n + \tfrac{1}{2})\pi, & n &= 0,1,2 \ldots \end{aligned} \tag{2.23}$$

The first set of solutions (2.23) correspond to bending waves which have been examined by Green[5]. The second set of solutions are symmetric about the middle plane of the plate and correspond to longitudinal waves travelling with speeds of propagation $v_n$ given by

$$v_n^2 = c_3^2 + \frac{(n+\frac{1}{2})^2\pi^2 c_1^2}{k^2h^2}, \quad n = 0,1,2 \ldots \tag{2.24}$$

The associated displacements and stresses are given by

$$u_1 = A_n \sin(n + \tfrac{1}{2})\pi \frac{x_1}{h} \cos\phi, \quad u_2 = 0, \quad u_3 = 0, \tag{2.25}$$

and

$$\begin{aligned} t_{11} &= \rho c_1^2 (n + \tfrac{1}{2}) \frac{\pi}{h} A_n \cos(n + \tfrac{1}{2})\pi \frac{x_1}{h} \cos\phi, \\ t_{33} &= -\rho c_3^2 (n + \tfrac{1}{2}) \frac{\pi}{h} A_n \cos(n + \tfrac{1}{2})\pi \frac{x_1}{h} \cos\phi \\ &\quad + \rho c_3^2 A_n (-1)^n \{\delta(x_1 - h) + \delta(x_1 + h)\} \cos\phi, \\ t_{13} &= -\rho c_3^2 (n + \tfrac{1}{2}) \frac{\pi}{ph} A_n \sin(n + \tfrac{1}{2})\pi \frac{x_1}{h} \sin\phi, \qquad |x_1| < h, \\ &= 0 \qquad\qquad |x_1| = h. \end{aligned} \tag{2.26}$$

Here $A_n$ is an arbitrary constant and $\delta(s)$, the Dirac delta function.

The fundamental mode of propagation is given by $n = 0$ and for this the phase velocity varies from the asymptotic value $c_3$ in the limit as $kh \to \infty$, increasing as $kh$

decreases and tending to infinity as $kh \to 0$. The associated displacements and stresses are obtained from (2.25) and (2.26) on putting $n = 0$. The wave number $k$ is related to the wavelength $\Lambda$ by the expression $k = 2\pi/\Lambda$ and the limits $kh \to \infty$ and $kh \to 0$ correspond to waves of vanishingly small and infinitely large wavelengths respectively. Introducing the circular frequency $\omega$ through the relation $\omega = kv$, eqn (2.24) may be rewritten in the form

$$v_n^2 = \frac{c_3^2}{1 - \{(n + \frac{1}{2})^2\pi^2 c_1^2\}/\omega^2 h^2}, \tag{2.27}$$

showing that the phase velocity $v_n$ has a cut-off at the frequency

$$\omega_n = \frac{(n + \frac{1}{2})\pi c_1}{h}, \tag{2.28}$$

and in particular the fundamental mode has a cut-off frequency of $\omega_0 = \pi c_1/2h$. The theory thus predicts that longitudinal waves with frequency lower than $\omega_0$ will not propagate in the inextensible plate.

## 3. ANISOTROPIC PLATE

In this section we examine wave propagation in a transversely isotropic plate without the inextensibility constraint. We again consider waves propagating in the direction of the axis of isotropy $Ox_3$, under plane strain conditions in the $x_1x_3$ plane. The appropriate constitutive equations are now (see Spencer[1])

$$\begin{aligned} t_{11} &= (\lambda + 2\mu_T)e_{11} + (\lambda + \alpha)e_{33} \\ t_{22} &= \lambda e_{11} + (\lambda + \alpha)e_{33} \\ t_{33} &= (\lambda + \alpha)e_{11} + (\lambda + 4\mu_L - 2\mu_T + 2\alpha + \beta)e_{33} \\ t_{13} &= 2\mu_L e_{13}, \end{aligned} \tag{3.1}$$

where $\alpha$ and $\beta$ are two additional material constants. Assuming displacements of the form

$$u_1 = U(x_1)\cos\phi, \quad u_2 = 0, \quad u_3 = W(x_1)\sin\phi, \tag{3.2}$$

the stresses are given by

$$\begin{aligned} t_{11} &= \rho(c_1^2 U' + c_4^2 kW)\cos\phi, \\ t_{22} &= \rho\{(c_1^2 - 2c_2^2)U' + c_4^2 kW\}\cos\phi, \\ t_{33} &= \rho(c_4^2 U' + c_5^2 kW)\cos\phi, \\ t_{13} &= \rho c_3^2(-kU + W')\sin\phi, \end{aligned} \tag{3.3}$$

where

$$c_2^2 = \frac{\mu_T}{\rho}, \quad c_4^2 = \frac{\lambda + \alpha}{\rho}, \quad c_5^2 = \frac{\lambda + 4\mu_L - 2\mu_T + 2\alpha + \beta}{\rho}. \tag{3.4}$$

Substituting from (3.3) into (2.8) the equations of motion become

$$\begin{aligned} c_1^2 U'' + k^2(v^2 - c_5^2)U + (c_3^2 + c_4^2)kW' &= 0, \\ -(c_3^2 + c_4^2)kU' + c_3^2 W'' + k^2(v^2 - c_5^2)W &= 0. \end{aligned} \tag{3.5}$$

For motions which are symmetric with respect to the middle plane of the plate, eqns (3.5) have the general solution

$$U = A_1 \sin kp_1x_1 + A_3 \sin kp_3x_1, \qquad (3.6)$$
$$W = -\frac{(c_1^2p_1^2 + c_3^2 - v^2)}{(c_3^2 + c_4^2)}\frac{A_1}{p_1}\cos kp_1x_1 - \frac{(c_1^2p_3^2 + c_3^2 - v^2)}{(c_3^2 + c_4^2)}\frac{A_3}{p_3}\cos kp_3x_1,$$

where $A_1$ and $A_3$ are arbitrary constants and $p_1^2$ and $p_3^2$ are, respectively, the algebraically greatest and least root of the quadratic

$$(p^2c_1^2 - v^2 + c_3^2)(p^2c_3^2 - v^2 + c_5^2) - (c_3^2 + c_4^2)^2p^2 = 0. \qquad (3.7)$$

Substituting the solutions (3.6) into the stress equations (3.3), the boundary conditions (2.9) are satisfied provided

$$\{c_1^2c_3^2p_1^2 - c_4^2(c_3^2 - v^2)\}\frac{A_1}{p_1}\cos kp_1h + \{c_1^2c_3^2p_3^2 - c_4^2(c_3^2 - v^2)\}\frac{A_3}{p_3}\cos kp_3h = 0,$$
$$(c_1^2p_1^2 - v^2 - c_4^2)A_1 \sin kp_1h + (c_1^2p_3^2 - v^2 - c_4^2)A_3 \sin kp_3h = 0. \qquad (3.8)$$

These equations have non-trivial solutions for $A_1$ and $A_3$ provided

$$\{c_1^2c_3^2p_1^2 - c_4^2(c_3^2 - v^2)\}(c_1^2p_3^2 - v^2 - c_4^2)\frac{\cos kp_1h}{p_1}\sin kp_3h$$
$$- \{c_1^2c_3^2p_3^2 - c_4^2(c_3^2 - v^2)\}(c_1^2p_1^2 - v^2 - c_4^2)\sin kp_1h\frac{\cos kp_3h}{p_3} = 0. \qquad (3.9)$$

Equation (3.9) is the secular equation which determines the phase velocity $v$ of longitudinal waves as a function of $kh$. The solution of this allows $A_1$ and $A_3$ to be expressed in terms of a single arbitrary constant $A$, using either of eqns (3.8), and leads to expressions for $U$ and $W$ in the form

$$U = A\left\{\frac{\sin kp_1x_1}{\sin kp_1h} - \frac{(c_1^2p_1^2 - v^2 - c_4^2)}{(c_1^2p_3^2 - v^2 - c_4^2)}\frac{\sin kp_3x_1}{\sin kp_3h}\right\},$$
$$W = -\frac{A}{(c_3^2 + c_4^2)}\left\{(c_1^2p_1^2 + c_3^2 - v^2)\frac{\cos kp_1x_1}{p_1 \sin kp_1h}\right. \qquad (3.10)$$
$$\left. - (c_1^2p_3^2 + c_3^2 - v^2)\frac{(c_1^2p_1^2 - v^2 - c_4^2)}{(c_1^2p_3^2 - v^2 - c_4^2)}\frac{\cos kp_3x_1}{p_3 \sin kp_3h}\right\}.$$

The expressions (3.10) when substituted into the last two of eqns (3.3) give equations for the stress components $t_{33}$ and $t_{13}$ in the form

$$\frac{t_{33}}{\rho kAc_3^2} = \frac{\cos kp_1x_1}{p_1 \sin kp_1h}\left\{-p_1^2 + \left(p_1^2 - \frac{v^2}{c_3^2}\right)\frac{(c_1^2p_1^2 + c_3^2 - v^2)}{(c_3^2 + c_4^2)}\right\}\cos\phi - \frac{\cos kp_3x_1}{p_3 \sin kp_3h}$$
$$\times\frac{(c_1^2p_1^2 - v^2 - c_4^2)}{(c_1^2p_3^2 - v^2 - c_4^2)}\left\{-p_3^2 + \left(p_3^2 - \frac{v^2}{c_3^2}\right)\frac{(c_1^2p_3^2 + c_3^2 - v^2)}{(c_3^2 + c_4^2)}\right\}\cos\phi,$$
$$\frac{t_{13}}{\rho kAc_3^2} = \frac{(c_1^2p_1^2 - v^2 - c_4^2)}{(c_3^2 + c_4^2)}\left\{\frac{\sin kp_1x_1}{\sin kp_1h} - \frac{\sin kp_3x_1}{\sin kp_3h}\right\}\sin\phi. \qquad (3.11)$$

To obtain the dispersion curve for phase velocity ($v$) as a function of scaled wave number ($kh$) for a specific material, it is necessary to solve the secular equation (3.9) numerically and this is done in Section 5. It is a relatively simple matter to solve eqn (3.9) for a general material in the limiting cases of infinitely long waves ($kh \to 0$) and infinitesimally short waves ($kh \to \infty$).

In the long wavelength limit, as $kh \to 0$, the sine functions in eqns (3.9) may be replaced by their arguments and the cosines have the value unity. Equation (3.9) then reduces to

$$\{c_1^4c_3^2p_1^2p_3^2 + c_1^2c_4^2(v^2 - c_3^2)(p_1^2 + p_3^2) - c_4^2(v^2 - c_3^2)(v^2 + c_4^2)\}\frac{(p_3^2 - p_1^2)}{p_1p_3} = 0. \quad (3.12)$$

From the quadratic equation (3.7), for $p^2$ we obtain expressions for $p_1^2p_3^2$ and $(p_1^2 + p_3^2)$ in the form

$$\begin{aligned} c_1^2c_3^2p_1^2p_3^2 &= (v^2 - c_3^2)(v^2 - c_3^2) \\ c_1^2c_3^2(p_1^2 + p_3^2) &= v^2(c_1^2 + c_3^2) + (c_3^2 + c_4^2)^2 - (c_1^2c_3^2 + c_3^4). \end{aligned} \quad (3.13)$$

Substituting from (3.13) into eqn (3.12) and eliminating non-zero factors leads to the equation

$$(v^2 - c_3^2)(c_1^2v^2 - c_1^2c_5^2 + c_4^4) = 0, \quad (3.14)$$

which has solutions

$$v^2 = c_3^2 \quad \text{or} \quad v^2 = c_5^2 - \frac{c_4^4}{c_1^2}. \quad (3.15)$$

The first of these solutions may be shown to be spurious and the second gives the limiting velocity of long waves in the fundamental longitudinal mode.

To examine the limiting solution of eqn (3.9) for short wavelengths we assume that $v^2 < c_3^2$ and both $p_1^2$ and $p_3^2$ are negative. Writing $p_1^2 = -q_1^2$, $p_3^2 = -q_3^2$ and dividing through eqns (3.9) by $\cos kp_1h \cos kp_3h$ leads to the equation

$$\{c_1^2c_3^2q_1^2 + c_4^2(c_3^2 - v^2)\}\{c_1^2q_3^2 + v^2 + c_4^2\}\frac{\tanh kq_3h}{q_1} - \{c_1^2c_3^2q_3^2 + c_4^2(c_3^2 - v^2)\}\{c_1^2q_1^2 + v^2 + c_4^2\}\frac{\tanh kq_1h}{q_3} = 0. \quad (3.16)$$

In the limit as $kh \to \infty$ the hyperbolic tangents in eqn (3.16) both have value unity and the equation reduces to

$$\frac{(q_3 - q_1)}{q_1q_3}\{c_1^4c_3^2q_1^2q_3^2 + c_1^2c_4^2(c_3^2 - v^2)(q_3^2 + q_1^2) - v^2c_1^2(c_3^2 + c_4^2)q_1q_3 + c_4^2(c_3^2 - v^2)(v^2 + c_4^2)\} = 0. \quad (3.17)$$

Using eqns (3.13), eqn (3.17) simplifies to the form

$$\left(c_5^2 - v^2 - \frac{c_4^4}{c_1^2}\right)\left(\frac{c_3^2 - v^2}{c_3^2 - v^2}\right)^{1/2} - \frac{c_3v^2}{c_1} = 0, \quad (3.18)$$

which is identical with the equation derived by Green[5] for the limiting speed of propagation of bending waves. The solution of eqn (3.18) gives the speed of propagation of the Rayleigh surface wave in the anisotropic material along the preferred direction.

## 4. STRONG ANISOTROPY AND INEXTENSIBLE LIMIT

The solutions derived in Section 3 are valid for all transversely isotropic materials. If the material is almost inextensible in the direction of transverse isotropy, we have

$$c_5^2 = c_3^2/\epsilon^2 \qquad (4.1)$$

where $\epsilon \ll 1$. In the limit as $\epsilon \to 0$, corresponding to $\beta \to \infty$, the material becomes the inextensible material considered in Section 2. Here we examine the secular equation (3.9) for the almost inextensible material. We consider solutions corresponding to the phase velocity $v$ remaining finite in the limits as $\epsilon \to 0$. Then eqn (3.7) has the form

$$(p^2c_1^2 - v^2 + c_3^2)\{(\epsilon^2p^2 + 1)c_3^2 - v^2\} - \epsilon^2(c_3^2 + c_4^2)^2p^2 = 0, \qquad (4.2)$$

with approximate solutions

$$p_1^2 = \frac{(v^2 - c_3^2)}{c_1^2} + 0(\epsilon^2), \quad p_3^2 = -\frac{1}{\epsilon^2} + 0(1). \qquad (4.3)$$

Substituting from (4.3) into (3.9) and using eqn (4.1) leads to the approximate secular equation

$$c_1(v^2 - c_3^2)^{1/2}\cos\left\{\frac{kh(v^2 - c_3^2)^{1/2}}{c_1}\right\}\sinh\left(\frac{kh}{\epsilon}\right) - \epsilon c_3^2\sin\left\{\frac{kh(v^2 - c_3^2)^{1/2}}{c_1}\right\}\cosh\left(\frac{kh}{\epsilon}\right) + O(\epsilon^2) = 0. \qquad (4.4)$$

for $v^2 > c_3^2$, eqn (4.4) has the approximate solution

$$v^2 = c_3^2 + \frac{c_1^2\pi^2}{4k^2h^2} - 2\epsilon c_3^2\left\{\frac{\coth(kh/\epsilon)}{kh}\right\} + 0(\epsilon^2), \qquad (4.5)$$

which in the limit as $\epsilon \to 0$ agrees with the solution (2.24) with $n = 0$. The solution (4.5) is not valid as $kh \to 0$ since this would indicate $v^2 \to \infty$ and violates the assumption leading to (4.3). The solution also breaks down as $kh \to \infty$ since this would lead to values of $v^2 < c_3^2$ and again violates the condition under which the approximate solution is obtained. In these two limits the behaviour of the phase velocity is given by eqns (3.15) and (3.18), respectively.

## 5. NUMERICAL RESULTS

The numerical solutions reported in this section are calculated using the experimental results of Markham[6] for a carbon-fibre-epoxy resin composite. For this material the squared wave speeds are given by

$$\rho c_1^2 = 10.57 \times 10^9\text{Nm}^{-2}, \quad \rho c_2^2 = 2.46 \times 10^9\text{Nm}^{-2}, \quad \rho c_3^2 = 5.66 \times 10^9\text{Nm}^{-2},$$
$$\rho c_4^2 = 4.37 \times 10^9\text{Nm}^{-2}, \quad \rho c_5^2 = 241.71 \times 10^9\text{Nm}^{-2},$$

and the corresponding value of $\epsilon$ is

$$\epsilon = \frac{c_3}{c_5} = 0.153.$$

These constants have been employed to determine the values of $kh$ corresponding to specified values of $v$ from eqn (3.9). The results are given in Table 1, together with

Table 1. Phase velocity as a function of wavelength, (a) Anisotropic material, exact solution, eqn (3.9), (b) Anisotropic material, approximate solution, eqn (4.5), (c) Inextensible material, eqn (2.24)

| | $v^2/c_3^2$ | | |
|---|---|---|---|
| kh | (a) | (b) | (c) |
| 0 | 42:386 | ∞ | ∞ |
| 0.255 | 42.000 | 70.355 | 71.641 |
| 0.321 | 40.000 | 44.626 | 45.607 |
| 0.356 | 35.000 | 36.563 | 37.440 |
| 0.387 | 30.000 | 30.965 | 31.766 |
| 0.426 | 25.000 | 25.703 | 26.427 |
| 0.478 | 20.000 | 20.532 | 21.175 |
| 0.688 | 10.000 | 10.276 | 10.721 |
| 1.958 | 2.000 | 2.046 | 2.202 |
| 3.349 | 1.300 | 1.320 | 1.411 |
| 5.210 | 1.100 | 1.111 | 1.170 |
| 11.882 | 1.000 | 1.007 | 1.033 |
| 18.861 | 0.990 | 0.997 | 1.013 |
| ∞ | 0.987 | 1.000 | 1.000 |

approximate values of $v$ calculated from (4.5). Also shown in Table 1 are the solutions for the fundamental mode of propagation of longitudinal waves in the inextensible material, obtained by taking $n = 0$ in eqn (2.24). It is clear that the approximate solution (4.5) has a limited range of validity and that the inextensible solution deviates significantly from the almost inextensible solution as $kh \to 0$.

The variation in shear stress $t_{13}$ and in the normal stress in the propagation direction $t_{33}$, through the plate thickness may be calculated using the solutions (3.11). This has been carried out for a number of values of $kh$ and the results are plotted in Figs. 1 and

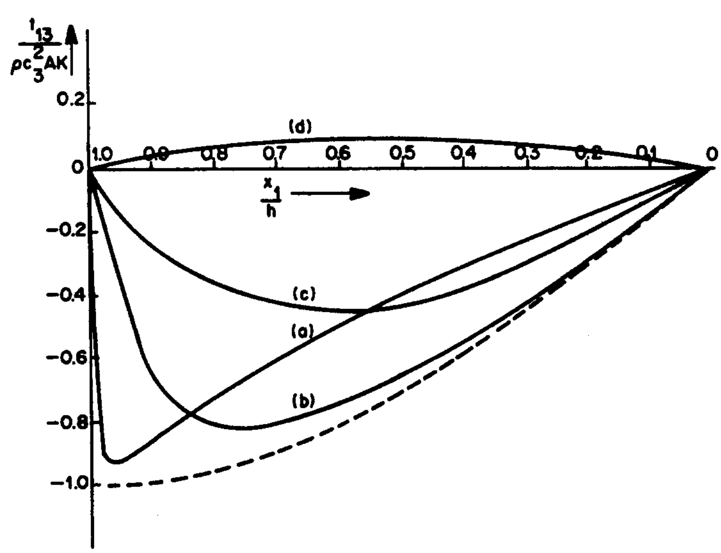


Fig. 1. Variation of shear stress $t_{13}$ through the plate for wavelengths given by, (a) $kh = 18.861$, (b) $kh = 1.958$, (c) $kh = 0.688$, (d) $kh = 0.255$. The dashed curve relates to the inextensible material.

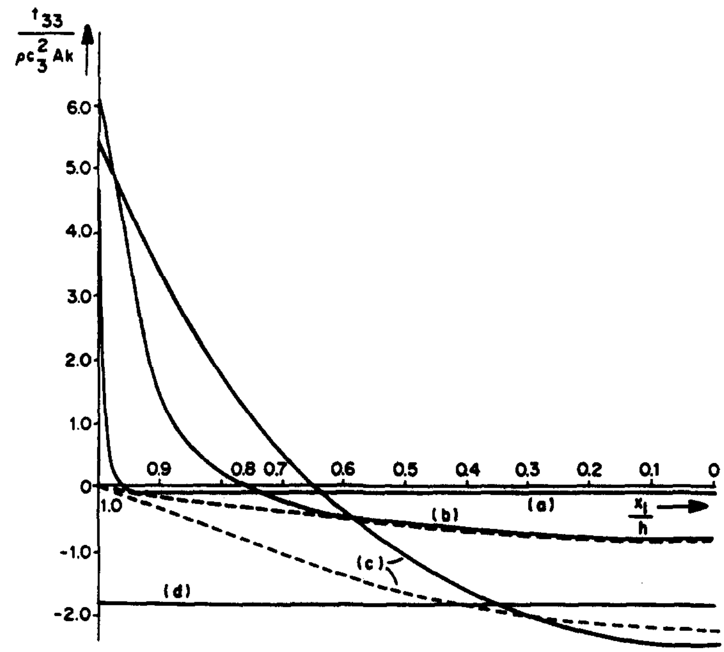


Fig. 2. Variation of shear stress $t_{33}$ through the plate for wavelengths given by, (a) $kh$ = 18.861, (b) $kh$ = 1.958, (c) $kh$ = 0.688, (d) $kh$ = 0.255. Scale must be multiplied by $10^2$ to get stress values for case (d). The dashed curves relate to the inextensible material and have a delta function singularity at $x_1/h$ = 1.

2, respectively. The graphs show the variation of stress over half the plate thickness, the abscissa running from the surface of the plate, $x_1/h = 1$, to the centre, $x_1/h = 0$. Also shown is the variation of stress through the plate for the fundamental longitudinal mode of propagation in the inextensible material. It should be noted that there is a delta function singularity in the normal stress $t_{33}$ at $x_1/h = 1$ which cannot be shown in the figure. It is clear from these figures that the stresses associated with waves in the inextensible plate are significantly different from those in the extensible plate in both the long-wave and short-wave limits.

## 6. DISCUSSION

The analysis of Section 4 shows that the solution for the inextensible material (Section 2) is the limiting case of the solution for the strongly anisotropic material (Section 3) provided $kh$ remains finite and non-zero in the limit as $\epsilon \to 0$. The portions of the dispersion curve corresponding to $kh \ll \epsilon$ and $kh \gg 1/\epsilon$ for the strongly anisotropic material are lost in the limit process which leads to the inextensible material. The stress states in the plate, shown in Figs. 1 and 2, clearly bring out the reason for this phenomenon. Thus, in the long-wavelength limit ($kh \ll \epsilon$) for the strongly anisotropic material, the state of stress through the plate thickness is a virtually constant normal stress in the propagation direction with practically zero shear stress, but it is not possible to reproduce this stress state in the inextensible material. It is for this reason that the inextensible material exhibits a cut-off frequency whereas longitudinal waves of all frequencies may propagate in the almost inextensible material. At the short-wave limit ($kh \gg 1/\epsilon$) both the shear stress and the normal stress in the almost inextensible material exhibit the exponential decay away from the plate boundaries which is associated with a surface wave, whilst the inextensible solution gives a completely different variation in shear stress through the plate thickness. The inextensible material therefore serves to model longitudinal-wave propagation in the fibre-reinforced material only for a restricted range of wavelengths and, in general, it is necessary to employ a transversely isotropic model which includes extensibility in the fibre direction. A similar conclusion arises from the analysis of flexural-wave propagation carried out by Green[5]. The results presented here refer specifically to waves propagating in the fibre direction, but an investigation of waves propagating at any angle to the fibre direction is currently being carried out.

*Acknowledgement*—We are grateful to Mr. R. P. Green for checking the numerical results. One of us (D.M.) is grateful to the Serbian Scientific Council for financial support.

## REFERENCES

1. A. J. M. Spencer, Continuum models of fibre-reinforced materials in *Mechanics of Structured Media* (Edited by A. P. S. Selvadurai) Elsevier, Amsterdam (1981).
2. A. H. England, J. E. Ferrier and J. N. Thomas, *J. Mech. Phys. Solids* **21**, 279 (1973).
3. L. W. Morland, *Int. J. Solids Structures* **9**, 1501 (1973).
4. A. C. Pipkin, Stress analysis for fiber-reinforced materials in *Adv. Appl. Mech.* **19**, 1 (1979).
5. W. A. Green, *Quart. J. Mech. Appl. Maths* **35**, 485 (1982).
6. M. F. Markham, *Composites* **1**, 145 (1970).